# A Multilevel Stochastic Collocation Method


Peter Jantsch

University of Tennessee, Knoxville


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Joint work with Aretha Teckentrup (FSU), Clayton Webster (ORNL/UT), and Max Gunzburger (FSU)



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# PDEs with Random Coefficients

Inputs to a mathematical model including computational domain, coefficients and source terms, may be subject to uncertainty due to:

- Incomplete or inaccurate knowledge
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**Goal:** Understand and propagate the effects of uncertainty to the output of the simulation.

## Model Problem - Linear Elliptic SPDE

Find $u \in L^2_\rho(\Gamma, H^1_0(D))$ such that for almost every $\mathbf{y} \in \Gamma$

$$\nabla \cdot (a(\mathbf{y}, x) \cdot \nabla u(\mathbf{y}, x)) = f(\mathbf{y}, x) \tag{1}$$

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Such a PDE might represent ground water flow, etc.

# Common Single Level Methods

## Monte Carlo Method

- Most popular method
- Simple to implement, easily parallelizable
- Convergence rate $\mathcal{O}(M^{-1/2})$ is dimension independant, but relatively slow

## Spectral Galerkin Methods

- Higher rate of convergence
- Degrees of freedom are coupled, leading to a large linear system
- Suffers from the curse of dimensionality

# Stochastic Collocation

For stochastic collocation we choose a set of (interpolatory) points $\{\mathbf{y}_j\}_{j=1}^{M} \subset \Gamma$, and for each $\mathbf{y}_j$ solve the deterministic PDE

$$\nabla \cdot (a(\mathbf{y}_j, x) \cdot \nabla u(\mathbf{y}_j, x)) = f(\mathbf{y}_j, x), \tag{2}$$

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For this scheme, we need to solve $M$ systems of size $n_h$. For high dimensional spaces $\Gamma$, the number of points $M$ needed to obtain a good approximation can be huge!

# History of the Multilevel Method

Multilevel methods for SPDEs derive from multigrid methods for the FEM, and have been used most commonly in the context of Monte Carlo methods:

- Multilevel Monte Carlo for numerical integration (S. Heinrich, 2001)
- Multilevel Monte Carlo path simulations for computational finance (M. Giles, 2008)
- Since applied to a variety of SPDEs

Main Idea: Suppose we have a sequence of finite element solutions $u_{h_k}(\mathbf{y}) \in V_{h_k}$, (with $u_{-1} = 0$). Multilevel methods are based on the following simple identity:

$$u_{h_K}(\mathbf{y}) = \sum_{k=0}^{K} u_{h_k}(\mathbf{y}) - u_{h_{k-1}}(\mathbf{y}).$$

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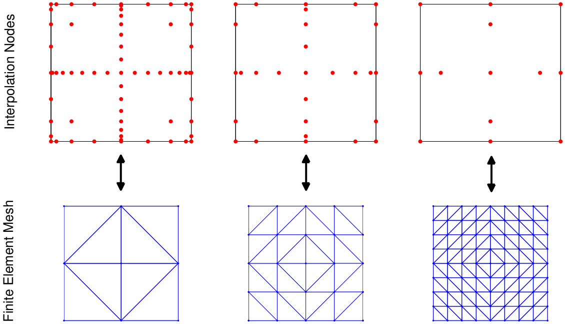

With Monte Carlo methods, we are usually interested in computing some statistics of the approximation $u_{h_K}(\mathbf{y})$. For instance, we can compute expectation using sample averages:

$$\mathbb{E}(u_{h_K}(\mathbf{y})) \approx u_{h_K}^{MLMC} = \sum_{k=0}^{K} \frac{1}{M_{K-k}} \sum_{j=1}^{M_{K-k}} \left( u_{h_k}(\mathbf{y}_j) - u_{h_{k-1}}(\mathbf{y}_j) \right). \quad (4)$$

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# Error Splitting

We examine the method by considering the discretization errors independently:

$$\begin{aligned} \|u - u_{h_K}^{ML}\| &\leq \|u - u_{h_K}\| + \|u_{h_K} - u_{h_K}^{ML}\| \\ &=: I + II. \end{aligned}$$

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The term $II$ can be further split apart using the triangle inequality:

$$\begin{aligned} II &= \| \sum_{k=0}^{K} (u_{h_k} - u_{h_{k-1}}) - \mathcal{I}_{l_{K-k}} (u_{h_k} - u_{h_{k-1}}) \| \\ &\leq \sum_{k=0}^{K} \| (\mathrm{I} - \mathcal{I}_{l_{K-k}})(u_{h_k} - u_{h_{k-1}}) \|. \end{aligned}$$

Now to compute the computational cost, we assume that the spatial discretization converges in $h$ as

$$I \leq C_s h_K^{\alpha},$$

and that the stochastic interpolation operators converge according to:

$$\|(\mathrm{I} - \mathcal{I}_{l_{K-k}})(u_{h_k} - u_{h_{k-1}})\| \leq C_I \, M_{K-k}^{-\mu} \, h_k^{\beta},$$

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Finally, we compute the cost of the multilevel method using the metric

$$Cost = \sum_{k=0}^{K} M_{K-k}\, C_k^{FEM} \approx \sum_{k=0}^{K} M_{K-k}\, h_k^{-\gamma}. \tag{6}$$

## Theorem: [J, Teckentrup, Webster, Gunzburger]

Under our assumptions, for any $\varepsilon > 0$ there exists an integer $K$ such that

$$\|u - u_{h_K}^{ML}\|_{L^2_\rho(\Gamma; H^1_0(D))} \leq \varepsilon$$

and

$$Cost_\varepsilon^{ML} \lesssim \begin{cases} \varepsilon^{-\frac{1}{\mu}} & \text{if } \beta > \mu\gamma, \\ \varepsilon^{-\frac{1}{\mu}} |\log \varepsilon|^{1+\frac{1}{\mu}} & \text{if } \beta = \mu\gamma, \\ \varepsilon^{-\frac{1}{\mu} - \frac{\gamma\mu - \beta}{\alpha\mu}} & \text{if } \beta < \mu\gamma. \end{cases} \tag{7}$$

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Compare to standard, single level SC:

$$Cost_\varepsilon^{SL} \eqsim h^{-\gamma} M \eqsim \varepsilon^{-\gamma/\alpha - 1/\mu}.$$

| Case | $\beta > \mu\gamma$ | $\beta = \mu\gamma$ | $\beta = \alpha < \mu\gamma$ |
|---|---|---|---|
| Cost Reduction | $\varepsilon^{-\gamma/\alpha}$ | $\eqsim \varepsilon^{-\gamma/\alpha}$ | $\varepsilon^{-1/\mu}$ |

# Example Problem:

As an example, we consider the following boundary value problem on either $D = (0,1)$ or $D = (0,1)^2$:

$$\begin{aligned} -\nabla \cdot (a(\omega, x)\nabla u(\omega, x)) &= 1, \qquad \text{for } x \in D, \\ u(\omega, x) &= 0, \qquad \text{for } x \in \partial D\,. \end{aligned}$$

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We take the coefficient $a$ to be of the form

$$a(\omega, x) = 0.5 + \exp\left[\sum_{n=1}^{N} \sqrt{\lambda_n} b_n(x) Y_n(\omega)\right],$$

where $\{Y_n\}_{n\in\mathbb{N}}$ is a sequence of independent, uniformly distributed random variables on [-1,1], and $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are the eigenvalues and eigenfunctions, resp., of the covariance operator with kernel function $C(x,y) = \exp[-\|x-y\|_1]$.

# Results in 10D

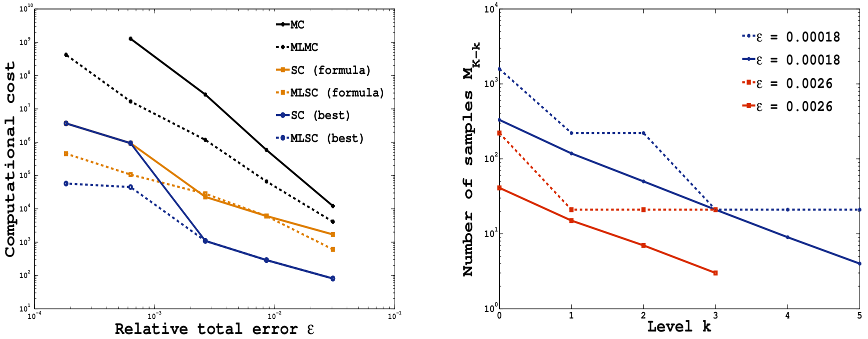
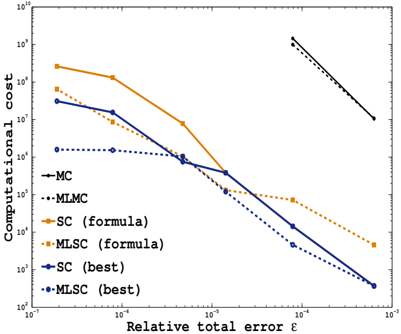



Figure : Left: Cost versus Error for $D = (0,1)^2$, $N = 10$. Right: Number of samples per level (predicted vs actual).

# Results in 20D



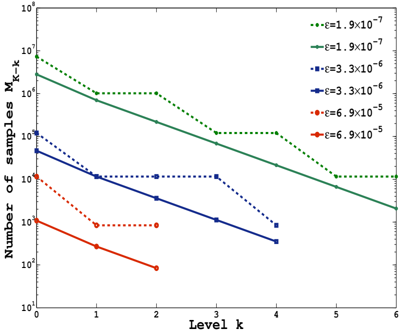




Figure : Left figures: Cost versus Error for $D = (0, 1)$, $N = 20$. Right figures: Number of samples per level (predicted vs actual).

# Multilevel methods:

- Can be practically applied to SC methods based on sparse grids
- Reduce computational cost for a variety of stochastic sampling methods for SPDEs.
- Work to counteract the curse of dimensionality.
- Effective when applied to SC schemes even up to 20D.